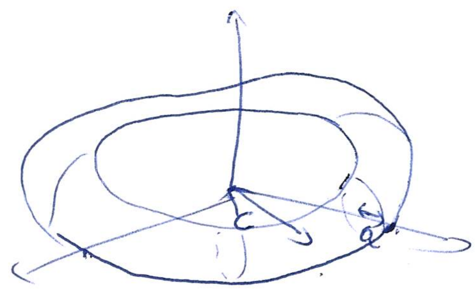
Another application of Gauss:

Ex: The torus T of inner radius a and outer radius c is parametrized by

$$(0 \leq u, v \leq 2\pi)$$

$$\Phi(u,v) = \big((c + a\cos v)\cos u, (c + a\cos v)\sin u, a\sin v\big)$$



Find the area it encloses.

By Gauss' theorem, $\displaystyle\iiint_{\text{Solid } T} \nabla\cdot\vec{F}\, dV = \iint_{T} \vec{F}\cdot d\vec{S}.$

So for any vector field such that $\nabla\cdot\vec{F} = 1$,

$$\text{Volume}(\text{Solid } T) = \iiint_{\text{Solid } T} 1\, dV = \iint_{T} \vec{F}\cdot d\vec{S}$$

Lots of choices of $\vec{F}$ ... Easiest: $\vec{F}(x,y,z) = (0,0,z)$

$$(\Rightarrow \nabla\cdot\vec{F} = 1)$$

Hence, $\mathrm{Vol}(\text{Solid } \Pi) = \iint_{\Pi} \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \cdot d\vec{S}$

Computing a normal vector:

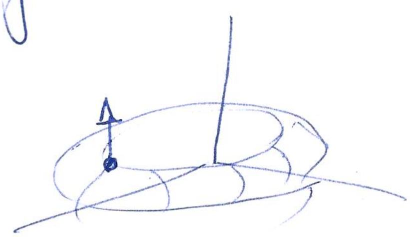
$$T_u = \begin{pmatrix} -(c+a\cos v)\sin u \\ (c+a\cos v)\cos u \\ 0 \end{pmatrix}, \quad T_v = \begin{pmatrix} -a\sin v\cos u \\ -a\sin v\sin u \\ a\cos v \end{pmatrix}$$

$$T_u \times T_v = a(c+a\cos v)\begin{pmatrix} \cos u\cos v \\ \sin u\cos v \\ \sin v \end{pmatrix}$$

Checking $T_u \times T_v$ points outwards: $(u,v) = (0, \frac{\pi}{2})$

$\Phi(0, \frac{\pi}{2}) = (c, 0, a)$

$T_u \times T_v(0, \frac{\pi}{2}) = (0, 0, ac)$



$\to$ OK!

$$\mathrm{Vol}(\text{Solid } \Pi) = \int_0^{2\pi}\int_0^{2\pi} \begin{pmatrix} 0 \\ 0 \\ a\sin v \end{pmatrix} \cdot a(c+a\cos v)\begin{pmatrix} \cos u\cos v \\ \sin u\cos v \\ \sin v \end{pmatrix} du\, dv$$

$$\mathrm{Vol}(\text{Solid } T) = \int_0^{2\pi}\int_0^{2\pi} a^2(c+a\cos v)\sin^2 v \, du\, dv$$

$$= 2\pi a^2 \int_0^{2\pi} \underbrace{c\sin^2 v}_{\text{To linearize}} + \underbrace{a\cos v \sin^2 v}_{\text{Easy to antiderive}} \, dv$$

$$= 2\pi a^2 \int_0^{2\pi} \frac{c}{2}\left(1-\cos(2v)\right) + a\cos v \sin^2 v \quad dv$$

$$= 2\pi a^2 \times 2\pi \times \frac{c}{2}$$

$$= \boxed{2\pi^2 a^2 c}$$

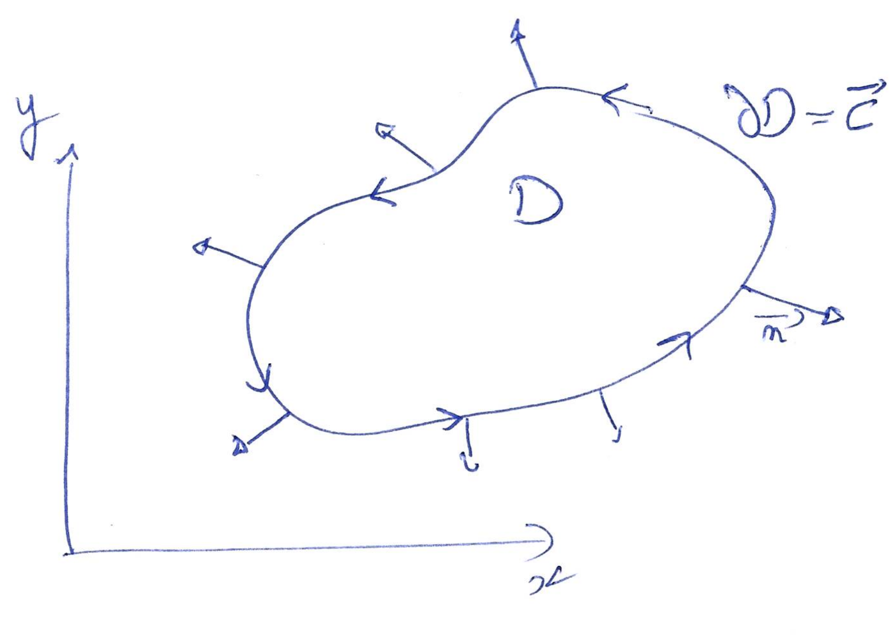
Rk: (Divergence version of Green's Theorem)

A two-dimensional version of Gauss is the following:

If $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$ (or equivalently $\vec{F} = \begin{pmatrix} P \\ Q \end{pmatrix}$), then for any region D bounded by closed curve $\vec{C}^+ = \partial D^+$,

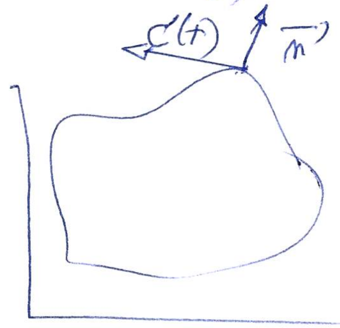
$$\boxed{\iint_D (\nabla \cdot \vec{F})\, dx\, dy = \int_{\vec{C}^+} \vec{F} \cdot \vec{n}\, ds},$$

where $\vec{n}$ is the unit (outward pointing) normal vector to ∂D



Application: Computing areas of domains!

- If ∂D is parametrized by $\vec{c}(t) = (x(t), y(t))$ with $a \leq t \leq b$ counter-clockwise orientation, then since $\vec{F}(x,y) = \begin{pmatrix} x \\ 0 \end{pmatrix}$ has divergence $\nabla \cdot \vec{F} = 1$,



$$\text{Area}(D) = \iint_D \nabla \cdot \vec{F}\, dx\, dy$$

$$= \int_{\vec{C}^+} \begin{pmatrix} x \\ 0 \end{pmatrix} \cdot \vec{n}\, ds$$

A unit normal vector to ∂D is $\vec{n}(t) = \dfrac{(+y'(t), -x'(t))}{\|\vec{c}'(t)\|}$,

$$\|\vec{c}'(t)\| \longrightarrow \sqrt{x'(t)^2 + y'(t)^2}$$

so
$$\boxed{\text{Area}(D)} = \int_a^b \begin{pmatrix} x(t) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} y'(t) \\ -x'(t) \end{pmatrix} \cdot \frac{1}{\|\vec{c}'(t)\|} \cdot \|\vec{c}'(t)\|\, dt$$

$$= \boxed{\int_a^b x(t) \cdot y'(t)\, dt}$$

- Similarly, taking $\vec{F}(x,y) = \begin{pmatrix} 0 \\ y \end{pmatrix}$, we get

$$\boxed{\text{Area}(D) = -\int_a^b y(t)\, x'(t)\, dt}$$

Rk: Back to Gauss (in $\mathbb{R}^3$), we get another way to see that

$$\nabla \cdot (\nabla \times \vec{F}) = 0 \quad \text{for all field } \vec{F}.$$

Indeed, for any solid W,

$$\iiint_W \nabla \cdot (\nabla \times \vec{F})\, dV \underset{\text{Gauss}}{=} \iint_{\partial W} (\nabla \times \vec{F}) \cdot d\vec{S}$$

$$\underset{\text{Stokes}}{=} \int_{\underbrace{\partial(\partial W)}_{\text{Empty!}}} \vec{F} \cdot d\vec{s}$$

$$= 0$$

Since this integral is zero for any domain W, the integrand $\nabla\cdot(\nabla\times\vec F)$ must be zero.

Review of Main Theorems

Theorem	Applies in	Applies to	States that
FTLI	2D & 3D	Curves / boundary of curve 	$\int_{\vec c} \nabla f\cdot d\vec s = f(\vec c(b)) - f(\vec c(a))$
Green	2D	Regions D / boundary of region ∂D 	$\iint_D (\nabla\times\vec F)\cdot\vec k\,dx\,dy = \int_{\partial D}\vec F\cdot d\vec s$ and $\iint_D (\nabla\cdot\vec F)\,dx\,dy = \int_{\partial D}\vec F\cdot\vec n\,ds$
Stokes	3D	Surfaces S / boundary of surfaces ∂S 	$\iint_S (\nabla\times\vec F)\cdot d\vec S = \int_{\partial S}\vec F\cdot d\vec s$
Gauss	3D	Regions W / boundary of regions ∂W 	$\iiint_W (\nabla\cdot\vec F)\,dx\,dy\,dz = \iint_{\partial W}\vec F\cdot d\vec S$